THEORY OF THE HEATING OF DIELECTRICS BY HIGH-POWER ELECTROMAGNETIC FIELDS

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The self-consistent problem of determining the interacting electromagnetic and temperature fields in a semi-infinite medium when a plane electromagnetic wave is incident normally on it is solved in the adiabatic approximation. The parameters of the medium are arbitrary functions of the temperature, thus enabling the WKB method to be used.

The development in the last decade of high-power microwave sources (with outputs of the order of tens and hundreds of kilowatts) has given rise to a new area of technology, which makes use of the intense heating of media with poor thermal conductivity by electromagnetic fields [1]. In such media the usual methods of thermal action are very inefficient.

Because of considerable mathematical difficulties, when calculating the electromagnetic and temperature fields in such media it is usually assumed that the physical parameters of the medium (the relative permittivity, the conductivity, the heat capacity, etc.) are independent of the temperature and, consequently, of the intensity of the electromagnetic field and the time for which it acts.

In many cases, however, the results of calculations of the field distribution in the medium differ considerably (by one or two orders of magnitude) from the experimental data. This is particularly so when the medium concerned is a semiconductor in which the conductivity depends exponentially on the temperature, or a medium in which a continuous phase transition is possible, for example, in media containing ice. The conversion of even a small amount of ice into water leads to a sharp change in the electrical parameters of the medium, since the permittivity of water exceeds that of ice and the remaining components of the medium by a factor of 10.

There is therefore a need to determine the electromagnetic and temperature fields in media whose parameters depend on the temperature. This is an extremely complex mathematical problem, requiring the simultaneous solution of Maxwell's equations and the equations of nonstationary thermal conductivity. In this case, to solve Maxwell's equations it is necessary to know the electrical parameters which depend on the temperature distribution in the medium, while the thermal-conductivity equation cannot be solved without knowing the power of the volume sources of heat, which are proportional to the divergence of Poynting's vector. Problems of this type are difficult to solve even numerically on modern computers.

Cases in which it is possible to obtain an analytical solution are therefore of particular interest. Such a case is the normal incidence of a plane electromagnetic wave on a semi-infinite medium in which the WKB method is used to calculate the electromagnetic field.

The WKB method can be used for media in which the electrical parameters vary over distances which considerably exceed the wavelength in the medium. This method can therefore be used for dielectrics with a low loss tangent, or for dielectrics in which the electrical parameters depend only slightly on the temperature. The parameters of the medium are arbitrary functions of the temperature. The time for which the heating can be assumed to be adiabatic is considered. This situation is the most interesting one for technical applications.

1. Fundamental Equations of the Problem. In the one-dimensional case Maxwell's equations take the form

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$$-\frac{\partial H}{\partial x}=\frac{\partial D}{\partial t}+j,\quad \frac{\partial E}{\partial x}=-\frac{\partial B}{\partial t}. \tag{1.1}$$

Here E is the electric field strength; H is the magnetic field strength; D is the electric induction; and B is the magnetic induction.

For a vacuum ($x < 0$): $j = 0$, $D = \varepsilon_0 E$, $B = \mu_0 H$, where ε_0 and μ_0 are the permittivity and permeability of free space.

For the medium ($x > 0$): $j = \sigma E$, $D = \varepsilon_0 \varepsilon E$, $B = \mu_0 H$. Here σ is the conductivity; and ε is the relative permittivity. The relative permeability of the medium is assumed to be equal to unity.

At the boundary of the medium ($x = 0$) the following continuity conditions are satisfied:

$$E(+0)=E(-0);\quad H(+0)=H(-0). \tag{1.2}$$

The electromagnetic wave incident on the medium has the form

$$E_0^+(x,\ t)=A_0\exp[i(\omega t-k_0x)],\quad H_0^+=E_0^+/2w_0,\quad S_0^+=A_0^2/2w_0, \tag{1.3}$$

where A_0 is the amplitude of the electric field; S_0 is Poynting's vector; w_0 is the intrinsic impedance of free space; k_0 is the wave number of the wave in vacuum; and ω is the angular frequency.

The temperature field in the medium in the adiabatic approximation is given by the equation

$$c\rho\frac{\partial T}{\partial t}=q(x,\ t);\ q(x,\ t)=-\operatorname{div}S=-\frac{1}{2}\operatorname{Re}\operatorname{div}(EH^*). \tag{1.4}$$

Here c and ρ are the heat capacity and density of the medium, respectively, which like σ and ε, are arbitrary functions of the temperature.

The boundary and initial conditions of Eq. (1.4) are

$$T(0,\ x)=T(t,\ \infty)=T_0. \tag{1.5}$$

The condition which must be satisfied in order that the adiabatic approximation can be used is

$$c\rho\frac{\partial T}{\partial t}\gg\lambda\left|\frac{\partial^2 T}{\partial x^2}\right|, \tag{1.6}$$

where λ is the thermal conductivity of the medium. Condition (1.6) is more conveniently written in the form

$$\frac{\partial T}{\partial t}\gg a^2\left|\frac{\partial^2 T}{\partial x^2}\right|, \tag{1.7}$$

where $a^2 = \lambda/c\rho$ is the thermal diffusivity of the medium.

The set of Eqs. (1.1)-(1.6) is the closed system of equations of the problem considered.

2. The Electromagnetic and Temperature Field in a Medium with Constant Parameters. We will briefly recall the basic equations for the electromagnetic and temperature fields in media whose parameters are independent of the temperature. The solution of Maxwell's equations has the form

$$\begin{gathered}E=A_0\{\exp[i(\omega t-k_0x)]+R\exp[i(\omega t+k_0x)]\},\\ H=\frac{A_0}{w_0}\{\exp[i(\omega t-k_0x)]-R\exp[i(\omega t+k_0x)]\}\quad(x<0),\\ E=A_0F\exp[i(\omega t-kx)],\quad H=E/w,\quad k=k_0\varepsilon_k^{1/2}\quad(x>0),\end{gathered} \tag{2.1}$$

where $w = (\mu_0/\varepsilon_0\varepsilon_k)^{1/2}$ is the intrinsic impedance of the medium; and ε_k is the complex permittivity of the medium ($\varepsilon_k = \varepsilon' - i\sigma/\omega = \varepsilon' - i\varepsilon''$).

The reflection coefficient with respect to the field R and the transmission coefficient with respect to the field F are

$$R=\frac{w-w_0}{w+w_0}=\frac{1-\varepsilon_k^{1/2}}{1+\varepsilon_k^{1/2}};\quad F=\frac{2w}{w_0+w}=\frac{2}{1+\varepsilon_k^{1/2}}.$$

Poynting's vector in the medium S(x) is

$$S(x) = F_e S_0^+ \exp(-2\alpha x), \tag{2.2}$$

where the energy-transmission coefficient F_e and the attenuation factor α are given by

$$F_e = \frac{2w_0(w + w^*)}{|w_0 + w|^2} = \frac{2(\varepsilon_k^{1/2} + \varepsilon_k^{*1/2})}{|1 + \varepsilon_k^{1/2}|^2}; \tag{2.3}$$

$$\alpha = \frac{k_0 \varepsilon'^{1/2}}{2^{1/2}} \operatorname{tg}\delta \,[1 + \sqrt{1 + \operatorname{tg}^2\delta}]^{-1/2}. \tag{2.4}$$

Here $\tan\delta = \varepsilon''/\varepsilon'$ is the dielectric loss tangent. The quantity $h = \alpha^{-1}$ is what is known as the skin depth.

Using Eq. (2.2), we obtain

$$q(x, t) = 2\alpha F_e S_0^+ \exp(-2\alpha x),$$

after which we find from Eq. (1.4)

$$T(x, t) = T_0 + \frac{2\alpha F_e S_0^+}{c\rho} t \exp(-2\alpha x). \tag{2.5}$$

It is seen from this equation that in the approximation considered the temperature is a linear function of time and of the power introduced.

Note that we mean by the skin depth of the temperature field the quantity h, since when x = h the temperature, measured from its initial value, decreases by a factor of exp (–2) with respect to its value at the boundary of the medium where x = 0.

The condition under which the adiabatic approximation (1.7) can be used gives

$$t \ll 0.25\,(h/a)^2, \tag{2.6}$$

whence we see that it is independent of the power introduced.

3. Solution of Maxwell's Equations for Media to Which the WKB Method Can be Applied. If the parameters of the medium are temperature dependent, i.e., they depend on the time and on the coordinates, it is not possible to solve Maxwell's equations in general form.

However, for high-frequency and microwave fields the time during which the electrical parameters of the medium vary is obviously much greater than the period of the oscillations, so that we can seek a solution of Maxwell's equations in the form

$$\begin{gathered} E = E_1(x, t)\exp(i\omega t), \quad H = H_1(x, t)\exp(i\omega t), \\ D(x, t) = \varepsilon_0\varepsilon(x, t)E, \quad B = \mu_0 H, \; j = \sigma(x, t)E, \; (x > 0), \\ D(x, t) = \varepsilon_0 E, \quad B = \mu_0 H, \quad (x < 0), \end{gathered} \tag{3.1}$$

where $E_1(x, t)$, $H_1(x, t)$, $\varepsilon(x, t)$, and $\sigma(x, t)$ are slowly varying functions of time. Substituting Eq. (3.1) into (1.1), and neglecting the derivatives with respect to time of slowly varying functions, we obtain

$$-\frac{dH_1}{dx} = i\omega\varepsilon_0\varepsilon_k(x, t)E_1, \quad \frac{dE_1}{dx} = -i\omega\mu_0 H_1. \tag{3.2}$$

In these equations the time plays the role of a parameter. Eliminating the magnetic field from Eqs. (3.2), we obtain an equation for the electric field,

$$\frac{d^2E_1}{dx^2} + k^2(x, t)E_1 = 0, \tag{3.3}$$

where

$$k^2 = k_0^2\varepsilon_k(x, t). \tag{3.4}$$

In Eq. (3.3) the law of variation of k(x, t) is unknown, so that it is difficult to obtain a solution of the equation in general form. But if this equation can be solved using the WKB method we have

$$E_1 = A_0 F\left[\frac{k(0, t)}{k(x, t)}\right]^{1/2} \exp\left(-i\int_0^x k(x, t)\,dx\right) + \frac{c}{\sqrt{k(x)}}\exp\left(i\int_0^x k(x, t)\,dx\right). \tag{3.5}$$

In Eq. (3.5), as shown in [2], we can neglect the second term, and interpret the first term as the "forward wave" with a transmission coefficient with respect to the field F

$$E_1 = A_0 F \left[\frac{k(0,\ t)}{k(x,\ t)}\right]^{1/2} \exp\left(-i\int_0^x k(x,\ t)\,dx\right). \tag{3.6}$$

With the assumed accuracy we have for the magnetic field in the medium

$$H_1(x,\ t) = E_1(x,\ t)/w \quad (x > 0). \tag{3.7}$$

The electromagnetic field in vacuum can be written in the form

$$E = A_0\{\exp[i(\omega t - k_0 x)] + R(t)\exp[i(\omega t + k_0 x)]\},$$
$$H = \frac{A_0}{w_0}\{\exp[i(\omega t - k_0 x)] - R(t)\exp[i(\omega t + k_0 x)]\} \ (x < 0), \tag{3.8}$$

where R(t) is the reflection coefficient with respect to the field and is a slowly varying function of time.

Using the boundary condition (1.2) we obtain

$$R(t) = \frac{w(0,\ t) - w_0}{w(0,\ t) + w_0} = \frac{1 - \varepsilon_k^{1/2}(0,\ t)}{1 + \varepsilon_k^{1/2}(0,\ t)};\quad F(t) = \frac{2}{1 + \varepsilon_k^{1/2}(0,\ t)}. \tag{3.9}$$

It follows from these equations that in the approximation assumed the transmission and reflection coefficients with respect to the field are determined by the value of the complex permittivity at the boundary of the medium.

4. Determination of the Temperature Field. Substituting the values of E_1 and H_1 from Eqs. (3.6) and (3.7) into Eq. (1.4), we obtain the following nonlinear integrodifferential equation:

$$c(T)\rho(T)\frac{\partial T}{\partial t} = M\{T(0,\ t)\}N\{T(x,\ t)\}\exp\left[-2\int_0^x \alpha(x,\ t)\,dx\right], \tag{4.1}$$
$$T(x,\ 0) = T(\infty,\ t) = T_0,$$

where

$$M\{T(0,\ t)\} = S_0^+ |F(t)|^2 |k(0,\ t)|/k_0,$$
$$N\{T(x,\ t)\} = \frac{w(x,\ t) + w^*(x,\ t)}{|w(x,\ t)|^2}\cdot\frac{k_0\alpha(x,\ t)}{|k(x,\ t)|}. \tag{4.2}$$

Equation (4.1) can be integrated in the following way.

We first determine the temperature at the boundary of the medium. It satisfies the following equation with separable variables:

$$c\{T(0,\ t)\}\rho\{T(0,\ t)\}\frac{\partial T(0,\ t)}{\partial t} = M\{T(0,\ t)\}N\{T(0,\ t)\},$$
$$T(0,\ 0) = T_0. \tag{4.3}$$

Since the temperature-dependence of all the parameters of the medium is assumed to be known, this equation is easily integrated,

$$t = \int_{T_0}^{T(0,t)} \frac{c(T)\rho(T)\,dT}{M\{T\}N\{T\}}. \tag{4.4}$$

Equation (4.4) gives the implicit time-dependence of the temperature of the boundary.

After finding T(0, t) the quantity M{T(0, t)} can be assumed to be a known function of time

$$M\{T(0,\ t)\} = M(t). \tag{4.5}$$

We replace the function and the variable in Eq. (4.1) using the relations

$$\tau = \int_0^t M(t)\,dt,\quad \Gamma(x,\ \tau) = \int_{T_0}^{T(x,\tau)} \frac{c(T)\rho(T)\,dT}{N\{T\}}. \tag{4.6}$$

Equation (4.1) then becomes

$$\frac{\partial \Gamma}{\partial \tau} = \exp\left[-2\int_0^x \alpha\{\Gamma(x, \tau)\}\,dx\right]. \tag{4.7}$$

We differentiate Eq. (4.7) with respect to x

$$\frac{\partial^2 \Gamma}{\partial x \partial \tau} = -2\alpha\{\Gamma\}\exp\left[-2\int_0^x \alpha\{\Gamma(x, t)\}\,dx\right]. \tag{4.8}$$

Hence, using relation (4.7), we obtain

$$\frac{\partial^2 \Gamma}{\partial x \partial \tau} + 2\alpha\{\Gamma\}\frac{\partial \Gamma}{\partial \tau} = 0 \tag{4.9}$$

or

$$\frac{\partial}{\partial \tau}\left[\frac{\partial \Gamma}{\partial x} + 2\int_0^\Gamma \alpha(\Gamma)\,d\Gamma\right] = 0,$$

and, consequently,

$$\frac{\partial \Gamma}{\partial x} + 2\int_0^\Gamma \alpha(\Gamma)\,d\Gamma = 0. \tag{4.10}$$

The constant of integration is assumed to be zero, since when $\tau = 0$, $\Gamma(0, x) = \partial\Gamma(0, x)/\partial x = 0$.

Equation (4.10) can easily be integrated to give

$$2x = \int_{\Gamma(x,\tau)}^{\Gamma(0,\tau)} \frac{d\Gamma'}{\int_0^{\Gamma'} \alpha(\Gamma'')\,d\Gamma''}. \tag{4.11}$$

Using the second of the relations (4.7), we obtain

$$2x = \int_{T(x,t)}^{T(0,t)} \frac{c(T')\rho(T')\,dT'}{N(T')\int_{T_0}^{T'} \frac{\alpha(T'')\rho(T'')c(T'')\,dT''}{N(T'')}}. \tag{4.12}$$

Equations (4.4) and (4.12) completely solve the problem, since, knowing the temperature distribution and using the given relations $\varepsilon = \varepsilon(T)$, $\delta(T)$ from Eqs. (3.6) and (3.7), we can determine the electromagnetic field in the medium.

For the important special case of constant density and heat capacity, Eqs. (4.4) and (4.12) can be simplified, and become

$$t = c\rho\int_{T_0}^{T(0,t)} \frac{dT'}{M\{T\}N\{T\}};\quad 2x = \int_{T(x,t)}^{T(0,t)} \frac{dT'}{N(T')\int_{T_0}^{T'} \frac{\alpha(T'')\,dT''}{N(T'')}}. \tag{4.13}$$

5. Heating of Media with Low Losses. If the dielectric loss tangent in the medium is much less than unity, the medium becomes a low-loss medium. The solutions obtained can then be simplified considerably.

From Eqs. (3.3), (3.4), and (4.2) we obtain to within terms that are quadratic in $\tan\delta$:

$$\alpha = \frac{k_0\varepsilon'^{1/2}\,\mathrm{tg}\,\delta}{2};\quad N\{T(x, t)\} = 2\alpha;$$

$$M\{T(0,\ t)\} = S_0^+ \frac{4\sqrt{\varepsilon'(0,\ t)}}{[1+\sqrt{\varepsilon'(0,\ t)}]^2} = S_0^+ F_e,$$
$$|k(x,\ t)| = k_0\sqrt{\varepsilon'(x,\ t)}; \quad |k(0,\ t)| = k_0\sqrt{\varepsilon'(0,\ t)};$$
$$F(0,\ t) = \frac{4}{[1+\sqrt{\varepsilon'(0,\ t)}]^2}. \tag{5.1}$$

Hence, Eqs. (4.13), which determine the temperature, take the form

$$2t = \int_{T_0}^{T(0,t)} \frac{c(T)\rho(T)\,dT}{S_0^+ F_e \alpha(T)}; \quad 2x = \int_{T(x,t)}^{T(0,t)} \frac{c(T)\rho(T)\,dT}{dT\int_{T_0}^{T} c(T')\rho(T'')\,dT'}. \tag{5.2}$$

For the case of constant heat capacity and density, these relations become

$$t = \frac{c\rho}{2S_0^+}\int_{T(x,t)}^{T(0,t)} \frac{dT}{F_e(T)\,\alpha(T)}; \quad 2x = \int_{T(x,t)}^{T(0,t)} \frac{dT}{(T-T_0)\,\alpha(T)}. \tag{5.3}$$

6. Example of the Calculation of the Temperature in a Low-Loss Dielectric. Consider a dielectric in which c, ρ, ε' are independent of the temperature, and $\tan\delta$, and, consequently, the attenuation factor α, is a linear function of the temperature

$$\alpha(T) = \alpha_0 + \gamma(T - T_0). \tag{6.1}$$

Since the power-transmission coefficient F_e in the case considered depends only on ε', and remains constant, the first of Eqs. (5.3) gives

$$t = \frac{c\rho}{2F_e S_0^+}\int_{T_0}^{T(0,t)} \frac{dT}{\alpha_0 + \gamma(T-T_0)} = \frac{0{,}5c\rho}{S_0^+ F_e \gamma}\ln\frac{\alpha_0+\gamma(T-T_0)}{\alpha_0},$$

whence

$$T(0,\ t) = T_0 + \frac{\alpha_0}{\gamma}[\exp(\beta t) - 1], \tag{6.2}$$

where

$$\beta = 2S_0^+ F_e \gamma / c\rho.$$

It is seen from Eq. (6.2) that the temperature at the boundary varies exponentially with the power introduced and the time, unlike the linear dependence when $\alpha = \alpha_0 =$ const. This equation well illustrates the incorrectness of neglecting the temperature dependence of the attenuation factor.

Using the second of Eqs. (5.3) we find the temperature T(x, t)

$$2x = \int_{T(x,t)}^{T(0,t)} \frac{dT}{(T-T_0)[\alpha_0+\gamma(T-T_0)]} = \frac{1}{\alpha_0}\ln\frac{T(0,\ t)-T_0}{\alpha_0+\gamma[T(0,\ t)-T_0]}\cdot\frac{\alpha_0+\gamma[T-T_0]}{T-T_0},$$

and, consequently,

$$T(x,\ t) = T_0 + \frac{\alpha_0}{\gamma}\cdot\frac{1-\exp(-\beta t)}{\exp(2\alpha_0 x) - [1-\exp(-\beta t)]}. \tag{6.3}$$

It follows from this equation that when $x \neq 0$ the temperature also depends exponentially on the time and the power, and the increase in the rate of rise of temperature for x close to zero leads to a slowing down in the increase in the temperature at the remaining points. As $t \to \infty$ and $x \neq 0$ the temperature approaches a finite limit, due obviously to the neglect of the thermal conductivity.

We will consider the problem of the depth of penetration of the fields. Following from Sec. 2, we define the penetration depth by the equation

$$T(h,\ t) - T_0 = e^{-2}[T(0,\ t) - T_0]. \tag{6.4}$$

Using Eq. (6.3), we obtain

$$h = 0.5h_0 \ln[1 + (e^2 - 1)\exp(-\beta t)], \tag{6.5}$$

where $h_0 = \alpha_0^{-1}$.

For small times the penetration depth varies linearly with time,

$$h = h_0[1 - 0.5(1 - e^{-2})\beta t]. \tag{6.6}$$

For large times we have

$$h = 0.5(e^2 - 1)h_0 \exp(-\beta t), \ (\beta > 0), \tag{6.7}$$

$$h = 0.5[\ln(e^2 - 1) - \beta t], \ (\beta < 0). \tag{6.8}$$

We will determine the distance and time for which it is permissible to neglect the thermal conductivity. Using Eq. (1.7), we find

$$0.25(h_0/a)^2 \gg \frac{1 - (1 - e^{-\beta t})e^{-2\alpha x}}{1 + (1 - e^{-\beta t})e^{-2\alpha x}} \cdot \frac{e^{\beta t} - 1}{\beta}. \tag{6.9}$$

This estimate can be simplified since the first factor on the right is of the order of unity when $\beta > 0$

$$0.25(h_0/a)^2 \gg \frac{\exp(\beta t) - 1}{\beta}. \tag{6.10}$$

As might have been expected, the condition under which the adiabatic approximation can be used now depends, unlike Eq. (3.6), on the power introduced and the derivative of the attenuation factor with respect to temperature.

In a later paper we will give the results of a calculation for media in which the parameters depend in a more complex way on the temperature.

LITERATURE CITED

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